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A Comparison of the Fourier–Hermite and Power Transform Solutions of the Nonlinear Vlasov Equation

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The Power Transform method of integrating the nonlinear Vlasov equation is applied to two physically significant problems and the solutions are compared with the results of the Fourier–Hermite method.

The problems considered are one dimensional, with periodic boundary conditions and involve (1) Landau damping in a Maxwellian plasma, and (2) the two-beam instability with equal electron beams.

The Power Transform method is discussed and several truncation techniques are presented. It is found that an extrapolation procedure can be used which allows a truncation of the infinite matrix without causing numerical instabilities in the nonlinear system. Close quantitative agreement between the results of the two methods is found.

I. INTRODUCTION

It has become increasingly apparent lately that computer simulation plays an indispensable role in the understanding of plasma phenomena. Although the dynamical equations of plasma physics are essentially Maxwell's equations and Newton's equations of motion, they are highly nonlinear and pose an impossible task for satisfactory analytical analysis; however, the gross macroscopic behavior of the plasma is quite simple to analyze through numerical simulation (see, e.g., "Methods in Computational Physics," Vol. 9).

Two fundamentally different formulations have emerged in the numerical simulation of plasmas. One method consists of numerically solving the Vlasov–Poissson set of equations while the other computes the dynamics of a large number of charged particles as they move in their self-consistent electric fields.

If a direct numerical simulation of the Vlasov equation is attempted, either the independent variable v must be transformed or the distribution function must be periodically reconstructed with appropriate weighting factors to suppress the fine structure in order to avoid a breakdown of the solution [1]. The Fourier–Fourier Transform method [2] incorporates a Fourier transformation of both phase space variables; whereas in the case of the Fourier–Hermite transformation method [3, 4], only the spatial variable is Fourier transformed and the velocity

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variable is represented by a series using Hermite polynomials. The major difficulty with the Fourier–Hermite method is the apparent numerical instability that arises when the infinite Hermite series is truncated. A large number of coefficients, 0(1000), must be kept in order to compute for times of interest if the infinite matrix is simply truncated. This results in the consumption of a great deal of computer time; thus, it is economically desirable to seek another truncation procedure.

Armstrong [4] has been able to slow down the increase in Hermite polynomials, and thus compute for a longer period of time, by introducing a collision term into the Fourier–Hermite code. This permits one to use a few less coefficients; however, the collision term does produce a noticeable change in the solution for the electric field [4]. With the Fourier–Hermite code in its present form there is as yet no satisfactory extrapolation scheme available to estimate accurately the coefficients beyond the boundary of the matrix.

A new transformation scheme, the Power Transform method, has recently appeared [5]. The purpose of this work is to examine critically the applicability of the Power Transform method to two physically significant problems and compare the results with those obtained by Armstrong [3, 4] with the Fourier–Hermite method. We will be concerned primarily with truncation procedures and the stability of the Power Transform method. We find that the Power Transform method is amenable to an extrapolation procedure which avoids the difficulty of a cutoff and yields an appreciable savings in computer time. Extremely close agreement is found with the Fourier–Hermite method.

The problems considered are one dimensional with periodic boundary conditions. The plasma is considered to be a collisionless electron gas; i.e., only the motions of the electrons will be considered and the system is macroscopically neutral with a uniform immobile positive background. Only electrostatic forces between the charges are considered. Solutions for both stable and unstable initial conditions are presented.

In the next section, we outline the Power Transform method with a discussion of the truncation procedures. In Section III, we present the solutions to the problems of Landau damping in a Maxwellian plasma and the two-beam instability.

II. The Power Transform Method

A. *A Review of the Power Transform Method*

We consider the Vlasov–Poisson set of equations in dimensionless units

$$\frac{\partial f(x, v, t)}{\partial t} + v \frac{\partial f(x, v, t)}{\partial x} - E \frac{\partial f(x, v, t)}{\partial t} = 0, \tag{1}$$

$$\frac{\partial E}{\partial x} = 1 - \int f(x, v, t)\, dv. \tag{2}$$

The details of the transformation are presented elsewhere [5] so we will outline only the procedure.

Fourier transforming the functions in both the spatial and velocity variables, we have

$$\frac{\partial F_n(y,t)}{\partial t} + k_0 n \frac{\partial F_n(y,t)}{\partial y} + \sum_{n=-\infty}^{+\infty} iE_q y F_{n-q}(y,t) = 0, \tag{3}$$

$$iE_q = \frac{1}{k_0 q} F_q(0,t). \tag{4}$$

Writing $F_n(y,t)$ as an expansion of powers of y, we obtain

$$F_n(y,t) = \sum_{\nu=0}^{\infty} a_{n,\nu}(t)\, g_\nu\, y^\nu e^{-y^2/2}, \tag{5}$$

where

$$g_\nu = 2^{\nu/2} \frac{\Gamma[(\nu+2)/2]}{\Gamma(\nu+1)}. \tag{6}$$

Substituting this expansion in Eq. (3) results in the following set of coupled equations:

$$\dot{a}_{n,\nu}(t) + k_0 n \gamma_\nu [(\nu+1)\, a_{n,\nu+1}(t) - \nu a_{n,\nu-1}] + i\nu\gamma_\nu \sum_{q=-\infty}^{+\infty} E_q(t)\, a_{n-q,\nu-1}(t) = 0 \tag{7}$$

with

$$iE_q(t) = -a_{q,0}(t)/k_0 q, \tag{8}$$

where

$$\gamma_\nu = g_{\nu+1}/g_\nu .$$

This set of equations is termed the "Power Transform method" [5]. Equation (7) was integrated forward in time on the IBM 360–65 computer at the University of Iowa by using a modified Adams–Bashford predictor–corrector algorithm.

Note that for symmetric initial conditions, as used in this work, the $a_{n,\nu}$ are all real, whereas the coefficients obtained from a Hermite expansion in velocity are alternately real and imaginary [4]. Joyce *et al.* [5] have shown that the two expansion schemes are identical except for a complex factor.

B. *Truncation Difficulties*

When the system of Eq. (7) is numerically integrated, we are faced with the same problem of truncating the ν index as with the Hermite index for the Fourier–Hermite method.

If we set $a_{n,\nu} \equiv 0$ for $\nu > N(N \equiv \nu_{\max})$, we are again forced to choose N very large to obtain correct macroscopic quantities for a reasonable length of time. This, of course, would provide no advantage over the Fourier–Hermite code. The present form of the linear Vlasov equations [Eq. (7)], however, lends itself to a polynomial extrapolation scheme whereby we can accurately estimate the $N + 1$ coefficient.

There is no reason to assume any regularity between the $a_{n,\nu}$; however, when a version of Eq. (7) corresponding to the linearized Vlasov equation is numerically integrated, the $a_{n,\nu}(\nu = 0, 1, 2, ...)$ form a very regular pattern for large nu for the linear system [5]. The $N + 1$ coefficient can, therefore, be determined by a polynomial extrapolation and thus close the system. With a fourth-order polynomial scheme, Joyce *et al.* [5] have shown that the linear Vlasov system can be reasonably represented by as few as 10 coefficients. We will discuss the effect the number of coefficients has on the solution for the electric field in Sections II-D and E.

For unstable initial conditions (the two-beam instability) we find that the coefficients no longer lie on a smooth curve but are somewhat scattered. Even with this scattering, we find reasonable agreement between the lower-order coefficients for N as small as 20. We find that by closing the system with the polynomial extrapolation procedure, the inaccuracies in describing the electric field are very slight even with $N \sim 0$ (20). If one wishes to examine the interaction of the waves with the distribution function, however, it is advisable to incorporate a larger number of coefficients.

C. Use of a Damping Term

If we modify the system of Eq. (7) in such a way that the amplitudes of the $a_{n,\nu}$ with ν close to N can never become large, then a truncation has the same effect as a reasonable guess of $a_{n,N+1}$. By adding a damping term similar to that used with the Fourier–Hermite code [4], to the right side of Eq. (7), we find that the coefficients with small ν are not affected by the damping of the $a_{n,\nu}$ which have large ν. The damping term we have chosen is

$$\left(\frac{\partial f}{\partial t}\right)_c = \left[\eta_c \left(\frac{\partial(v)}{\partial v} + \frac{\partial^2}{\partial v^2}\right)\right]^l f$$

which assumes the form

$$(\delta a_{n,\nu}/\delta t)_c = -(\eta_c \nu)^l \, a_{n,\nu}(t).$$

With a collision term of this form, the coefficients can be written as

$$a_{n,\nu} = Ce^{-(\eta_c \nu)^l t},$$

with $\eta_c \ll 1$. The coefficients with small ν are only slightly affected by the collision

term since $\nu\eta_c \ll 1$; thus $(\nu\eta_c)^l \ll (\nu\eta_c)$; however, the coefficients with ν large are heavily damped since $\nu\eta_c \gg 1$ and $(\nu\eta_c)^l \gg \nu\eta_c$. This collision term thus selectively suppresses only the large v derivatives of the distribution function.

We find that for $(\eta_c N) \sim 0$ (1), the instability that appears in the Landau-damped case is suppressed without adding noticeably to the damping decrement; and we obtain excellent results for N as small as 80. For the two-stream instability problem, it is necessary to keep $(\eta_c N) < 1$ so that the macroscopic quantities are not significantly affected.

We will now discuss the results of these cutoff procedures in more detail with specific examples for both the stable and unstable initial conditions. In Section III-A we discuss the stable initial condition (the simple Maxwellian distribution); and in Section III-B we present the results for the unstable case (the two-beam instability).

III. Numerical Results

A. *Numerical Results for Stable Initial Conditions*

In all of the cases to be considered in this section, the form of the initial distribution is the Maxwellian distribution

$$f_0(v, 0) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}.$$

We found early in the calculations that $|E_2(t)|$ was at least an order of magnitude smaller than $|E_1(t)|$; thus, only the results with $n = 1$ will be presented.

We will consider wave numbers of $k = 0.3$, 0.4, 0.5, and 0.9 with a perturbation $\epsilon = 0.035$. We will first present the results of the extrapolation scheme. We find that we can easily avoid the difficulty of a cutoff by accurately estimating the $N + 1$ coefficient and we are able to describe the system for a reasonable length of time with as few as 10 coefficients. We will then discuss the results obtained by using the collision term to avoid an early breakdown of the system.

As an illustration of the results, we have chosen $k = 0.5$ and $\epsilon = 0.035$. Keeping only 80 coefficients, we then closed the set of equations by using a fourth-order polynomial extrapolation scheme. This allows us to estimate accurately the $N + 1$ coefficient. No breakdown of the system appears for the total time computed $(50\,\omega_p^{-1})$. As shown in Fig. 1, the electric field undergoes linear decay with a damping decrement of $\gamma_1 = -0.158$ until $t = 28\,\omega_p^{-1}$ when the decrement falls below the linear value. The damping decrement for $t < 28\,\omega_p^{-1}$ agrees exactly with the linear theoretical calculations done by Gary [6]. The results also agree favorably with the work done by Armstrong [3, 4] where the decrease in the damping decrement appears at almost the same time.

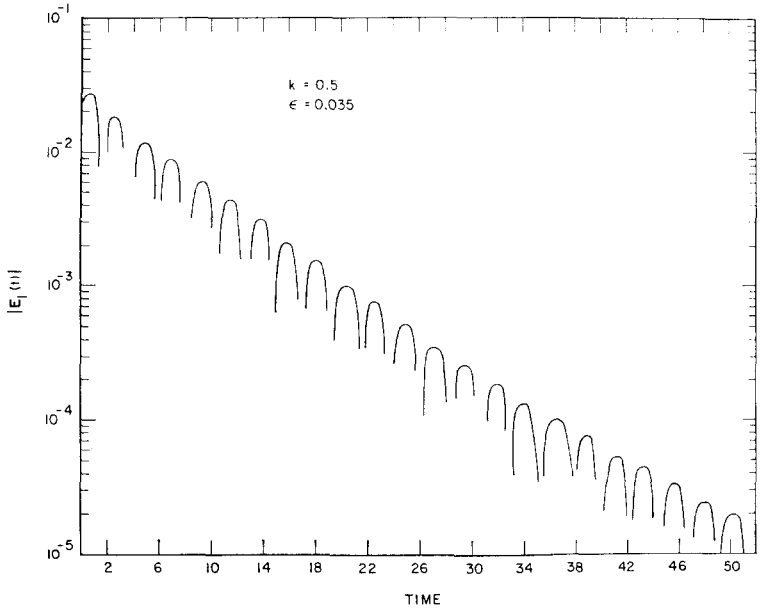

FIG. 1. Numerical integration of the first-order system for $k = 0.5$, $\epsilon = 0.035$, $N = 80$, for stable initial conditions using the extrapolation scheme.

As noted previously, if we truncate the system with a small number of coefficients, we can suppress the large v derivatives by adding a damping term of the form $-(\eta_c \nu)^l a_{n,\nu}(t)$ to the right side of Eq. (7). With 80 coefficients and $\eta_c = 0.016$, $l = 4$, we find that the breakdown of the system is completely suppressed; however, the field now undergoes strictly linear decay for the total time of computation. As the results indicate, this damping term selectively suppresses only the higher-order coefficients without adding significantly to the damping decrement for $t < 30\ \omega_p^{-1}$. We do note that for $t > 30\ \omega_p^{-1}$, the collision term adds slightly to the damping decrement obtained with the extrapolated code.

The breakdown of the system manifests itself through a large jump in the size of the electric field. With $k = 0.5$ and $\epsilon = 0.035$ and truncating the matrix at 80 coefficients, we find that $|E_1(t)|$ undergoes linear decay until $t = 32\ \omega_p^{-1}$ when breakdown becomes apparent. We note that the sharp growth in the electric field appears at $t = 2\sqrt{N}/nk_0$. This corresponds to the predicted time at which the ν-th coefficient attains its maximum value, $t = \sqrt{\nu}/nk_0$ [3].

By choosing N larger, the instability appears at later times; and with $N \sim 0(250)$, the solution is in exact agreement with the extrapolated solution with only 80 coefficients.

A summary of the results obtained with the four truncation schemes, i.e., (a)

TABLE I

Summary of Results for Landau Damping of the Maxwellian Distribution

Scheme[a]	N[b]	η[c]	T'[d]	Deviation[e]
A	80	—	—	—
B	80	—	—	0
C	80	0.0062	—	0
D	80	0.0062	—	0
		$k = 0.3; \epsilon = 0.035$		
A	80	—	—	—
B	80	—	35	*bd*
C	100	0.0062	43	+120
C	80	0.01	—	0
D	80	0.0062	45	−0.2
		$k = 0.4; \epsilon = 0.035$		
A	80	—	—	—
A	40	—	30	−25
A	20	—	29	−31
A	10	—	25	−19
B	80	—	31	*bd*
B	250	—	—	0
C	80	0.0062	33	*bd*
C	80	0.010	33	+43
C	80	0.016	32	−30
D	80	0.0062	48	−0.1
D	80	0.016	32	−30
		$k = 0.5; \epsilon = 0.035$		
A	80	—	—	—
B	80	—	19	*bd*
C	100	0.0062	21	*bd*
C	100	0.013	21	+32
D	80	0.0062	18	−0.1
		$k = 0.9; \epsilon = 0.035$		

[a] Truncation scheme: A: Polynomial extrapolation. B: Truncation; no collision term. C: Truncation with collision term. D: Extrapolation with collision term.

[b] N = total number of coefficients.

[c] η = collision term.

[d] T' = time (in ω_p^{-1}) at which first variations appeared.

[e] Deviation. Average percentage of deviation from extrapolated solution; *bd* = breakdown (manifested by large regrowth of E).

extrapolation with a fourth-order polynomial, (b) pure truncation of the matrix, (c) truncation with the addition of a collision term, and (d) extrapolation with a collision term, is presented in Table I. In all runs, a pure truncation of the matrix results in a complete breakdown of the system after a short period of time. Only in the case of $k = 0.4$ were we able to suppress the breakdown of the truncated system with a collision term without adding to the Landau damping.

We find that the extrapolation scheme can estimate the $N + 1$ coefficient to such a high degree of accuracy that the addition of a collision term is unnecessary. In all trials, the results of the fourth-order polynomial extrapolation scheme are in excellent agreement with the Fourier–Hermite solutions obtained by Armstrong [3, 4].

We now further investigate the extrapolated Power Transform method by examining the change in $| E_1(t)|$ as we reduce the number of coefficients. We choose the $k = 0.5$ and $\epsilon = 0.035$. We find that the results are in exact agreement for a reasonable length of time with N as small as 10. We have plotted the envelope of $| E_1(t)|$ for $N = 80$, 40, 20, and 10 in Fig. 2. Only with $N = 10$ is there any change in $| E_1(t)|$ before $t = 30\ \omega_p^{-1}$.

We note that, as the number of coefficients is decreased, the damping rate approaches its linear value. With 20 coefficients the field undergoes strict Landau

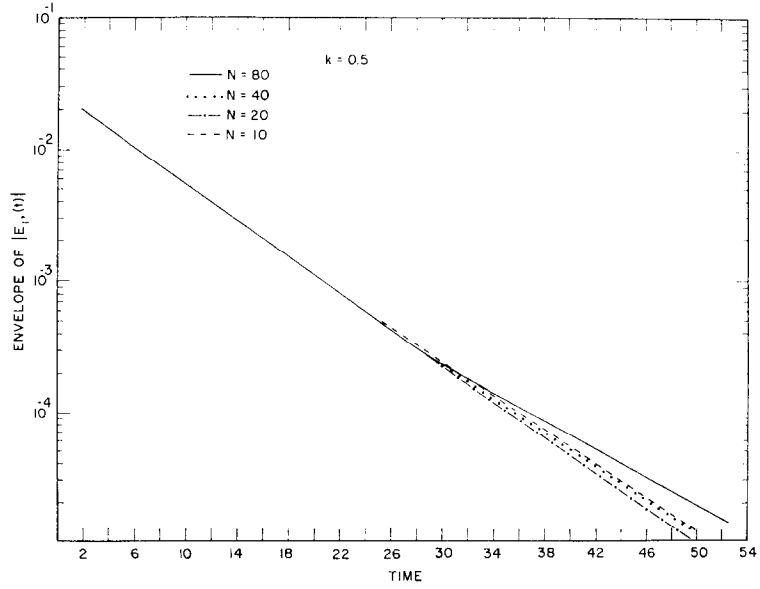

FIG. 2. Plot of the envelope of $| E_1(t)|$ for maximum $\nu = 10$, 20, 40, 80, for the extrapolated scheme with $k = 0.5$ and $\epsilon = 0.035$.

damping with $\gamma_1 = -0.158$ for the total time computed. We note that a similar increase in the damping rate was found for very large collision frequencies.

As the solutions for stable initial conditions quite clearly show, we are able to circumvent the difficulty of a cutoff mechanism by using an extrapolation scheme with the Power Transform method. This scheme is capable of estimating the $N + 1$ coefficient to such a degree that relatively few coefficients are needed in order to obtain excellent results.

We now want to apply the Power Transform method to the solution of unstable initial conditions. We find that the results are again in excellent agreement with the Fourier–Hermite Transform method.

B. *Numerical Results for Unstable Initial Conditions*

We will now investigate the results of the three cutoff procedures on the unstable initial condition, i.e.,

$$f_0(v, 0) = (1/2\pi)\, v^2 e^{-v^2/2}.$$

We found for the unstable case that we could not obtain reasonable results by just truncating the matrix at $N = 80$. The coefficients grew too large, resulting in

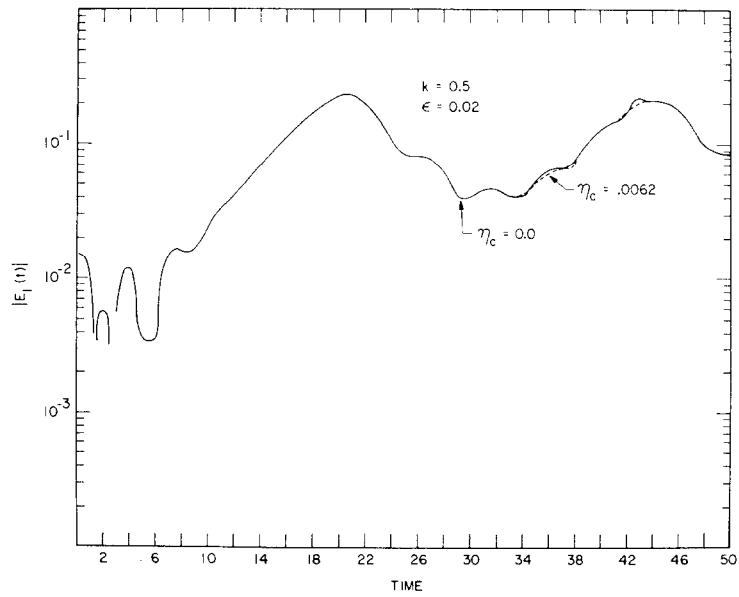

FIG. 3. Comparison of the extrapolated and the extrapolated with collisions solutions of the unstable case, $k = 0.5$, $\epsilon = 0.02$, $N = 80$, $\eta_c = 0.0062$, $l = 4$.

early failure of the program. By using a small damping term ($\eta_c \sim 0.0062$), however, we are able to obtain somewhat reasonable results for times of interest. We had no difficulty with the extrapolated code.

For the first example we have chosen $k = 0.5$ and $\epsilon = 0.02$ with $n = 1$. The solution of $E_1(t)$ for the fourth-order polynomial extrapolation scheme with $N = 80$ is presented in Fig. 3. $E_1(t)$ was observed to grow exponentially with a growth rate of $\gamma_1 = 0.245$. The growth ceased at $t = 20.5\ \omega_p^{-1}$, after which the field undergoes a long period of fluctuation in magnitude. Very small fluctuations appear in the electric field between $t = 34$ and $t = 44$. These ripples can be removed by adding a very small damping term ($\eta_c = 0.0062$) to the extrapolated code. This is also shown in Fig. 3.

It is thought that this roughness in $E_1(t)$ is due more to the fact that we are solving the $n = 1$ solution only and not to inaccuracies in the extrapolation procedure. We find that these ripples do not appear for $n \geqslant 2$. It is to be noted that the growth rate for $k = 0.5$ ($\gamma_1 = 0.245$) is in exact agreement with Armstrong's results [3]. Thus, it appears that in a situation where computer time is at a premium, a small collision term could be used with the extrapolated code as a means of obtaining excellent results with only the first harmonic.

With the same wave number and a perturbation $\epsilon = 0.04$, we calculated the

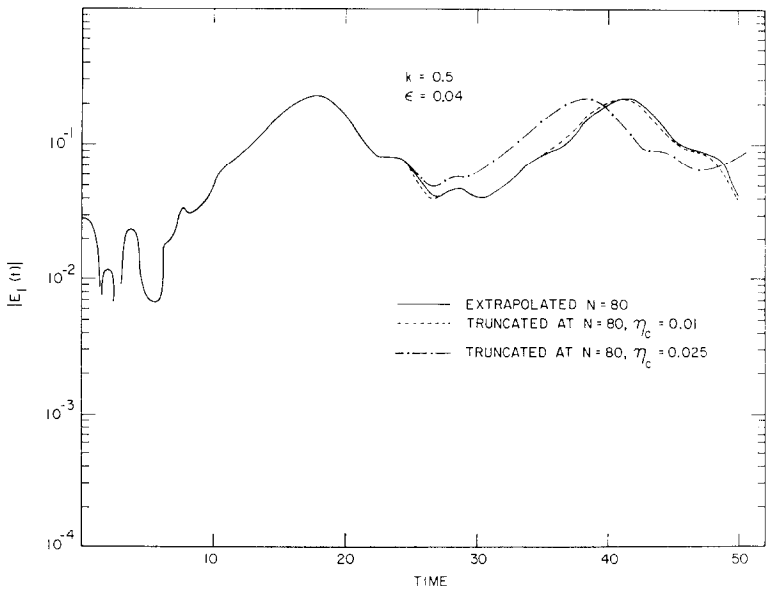

FIG. 4. Comparison of the extrapolated and damped truncated solutions for the unstable case, $N = 80$, $\eta_c = 0.01$ and 0.025 with $l = 4$, $k = 0.5$, $\epsilon = 0.04$.

solution of $E_1(t)$ by using the fourth-order extrapolation scheme. We find that the extrapolated solution is in very close agreement with the truncated solution with $\eta_c = 0.01$. This is shown in Fig. 4 where we have superposed the two solutions. We also show in this figure the early regrowth of $E_1(t)$ that comes about if we set $\eta_c > 0.01$. At no time did the damping term change the solution of the electric field before a time $t \sim 0(25)\,\omega_p^{-1}$. This is in contrast to Armstrong's collision term which affects the magnitude of the field as early as $t \sim 0(8)\,\omega_p^{-1}$ [4].

For a third example we have $k = 0.5$, $\epsilon = 0.2$, and $n = 2$. The solution is shown in Fig. 5. $E_1(t)$ undergoes exponential growth after $t = 10\,\omega_p^{-1}$ with a growth rate of $\gamma_1 = 0.245$. The field saturates at $t = 22\,\omega_p^{-1}$ after which it undergoes a smooth fluctuation in magnitude until computation was stopped at $t = 50$. $E_2(t)$ grows at a rate of $\gamma_2 \sim 2\gamma_1$ until it reaches a maximum at $t = 18.75\,\omega_p^{-1}$, after which it undergoes a period of fluctuations in magnitude with changes in sign. These results agree very favorably with Armstrong's results [4].

By retaining the first two modes, we have a smoother regrowth of $E_1(t)$ and the strictly exponential growth of the first mode ceases at an earlier time than for $n = 1$, although it reaches its saturation point at a slightly larger time. These differences are due to the k wave feeding energy into the $2k$ wave through the $E_1(\partial f_1/\partial v)$ term.

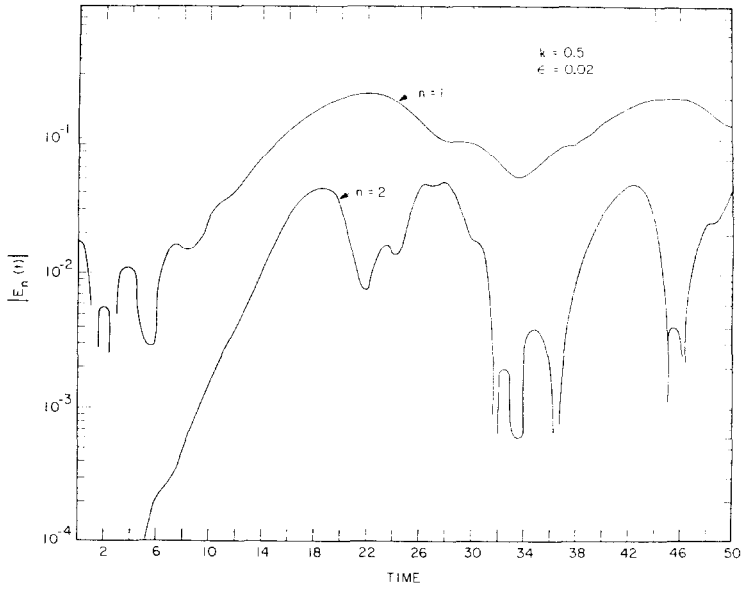

FIG. 5. Numerical integration of the second-order system for unstable initial conditions, using the extrapolation scheme, with $k = 0.5$, $\epsilon = 0.02$, $N = 80$.

TABLE II

Summary of Results for Two-Beam Instability

Scheme	N	η	T'	Deviation
A	80	—	—	—
A	40	—	32	9.7
A	20	—	27.5	35
B	80	—	22	*bd*
C	80	0.0062	33	3.9
C	80	0.01	32.5	11.6
C	80	0.025	27.5	40.3
D	80	0.0062	33	1.2
		$k = 0.5; n = 1; \epsilon = 0.02$		
A	80	—	—	—
B	80	—	11	*bd*
C	80	0.0062	32	1.3
C	80	0.01	24	6.1
C	80	0.025	24	39
		$k = 0.5; n = 2; \epsilon = 0.04$		

A summary of these solutions plus the results for several other runs are presented in Table II. As the numerical results clearly indicate, the extrapolated Power Transform method is equally applicable to the solution of unstable initial conditions.

As we stated earlier (Section II), there is some scattering among the higher-order coefficients for the unstable case. We want to investigate the solutions of the electric field as we reduce the number of coefficients in the extrapolated scheme. We obtain reasonable results for the total time computed with N as small as 20.

The solution of E_1 is shown in Fig. 6 for $k = 0.5$ with $N = 80$, 40, and 20. The solutions with $N = 80$ and 40 are remarkably close, and with $N = 20$ there is only a slight shift in the regrowth of the field. We note that the error introduced by using a small number of coefficients is similar to the error resulting from suppressing the higher-order coefficients by a large collision term (recall Fig. 4).

Summary of Results

The incorporation of a fourth-order polynomial extrapolation scheme into the Power Transform method has provided us with a rapid and efficient means of solving the nonlinear Vlasov equation for two important initial conditions. We find that we are able to avoid the difficulty of a cutoff. The numerical results indicate that this system is somewhat more stable numerically than the Fourier–Hermite code; and the inclusion of a damping term in the truncated scheme does not influence the solution to as large a degree as with the Fourier–Hermite code.

We find that the extrapolation procedure accurately estimates the $N + 1$ coefficient to such an extent that less than 80 coefficients are needed to describe the system for times $t \sim 0(50\,\omega_p^{-1})$. We note that for the unstable case, if we use only 20 coefficients the regrowth of E_1 is shifted by only $\Delta t = 5\omega_p^{-1}$ (see Fig. 6). Comparing this with Armstrong's results [4], we see that with a collision frequency as small as $\eta_c = 0.001$, variations appear in the electric field as early as $t = 8\,\omega_p^{-1}$ and the regrowth is shifted by a $\Delta t = 4\omega_p^{-1}$. Incorporating very few coefficients in the solution has not affected the limiting amplitude of the field nor has it changed the asymptotic behavior.

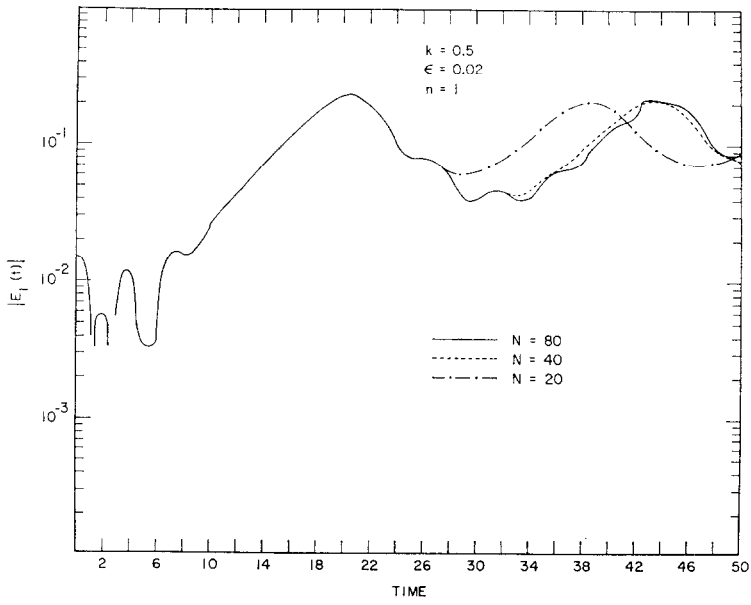

FIG. 6. Comparison of the extrapolated solutions for $|E_1(t)|$ for the two-beam instability, with v maximum = 20, 40, 80, with $k = 0.5$, $\epsilon = 0.02$, $n = 1$.

An example of the savings in computer time; the Fourier–Hermite code requires approximately $\frac{1}{2}$hr computer time for unstable initial conditions, whereas the extrapolated Power Transform method takes 2 min with 80 coefficients and only 17 sec with $N = 20$.

Acknowledgments

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